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# Repeated Angles in the Plane and Related Problems*

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We show that a set of $n$ points in the plane determine $O(n^2 \log n)$ triples that define the same angle $\alpha$, and that for many angles $\alpha$ (including $\pi/2$) this bound is tight in the worst case. We also show that, for a broad family of properties $\mathscr{P}$, the number of triangles spanned by the given points and having property $\mathscr{P}$ is $O(n^{7/3})$. Typical such properties are: having a specified area, a specified perimeter, being isosceles, etc. 

## 1. INTRODUCTION

Forty five years ago Paul Erdős [9] initiated the investigation of the distribution of the $\binom{n}{2}$ distances determined by $n$ points in the plane. In particular, he asked what is the maximum number of times that the same distance can occur among $n$ points. This question and the analogous problem in higher dimensions stimulated a lot of research and was the sub-

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ject of many papers in the last four decades. (See [10–13, 18, 3, 2, 21, 4].) Nevertheless, for most of these problems there is still a large gap between the best known lower and upper bounds.

In this paper we shall address the related problem concerning the distribution of the $3\binom{n}{3}$ angles determined by the triples of an $n$-element point set. Let $f^{(d)}(n, \alpha)$ denote the maximum number of times that the angle $\alpha$ can occur among the ordered triples of $n$ points in euclidean $d$-space, and let

$$f^{(d)}(n) = \max_{0<\alpha<\pi} f^{(d)}(n, \alpha).$$

It is known that $f^{(2)}(n) = o(n^{5/2})$, $f^{(3)}(n) = o(n^3)$ [19, 7, 6]. However, $f^{(4)}(n, \pi/2) \geqslant (\frac{1}{27} + o(1))\, n^3$, but $f^{(4)}(n, \alpha) = O(n^{3-1/5})$ if $\alpha \neq \pi/2$ [20].

The main result of this paper is that $f^{(2)}(n) = O(n^2 \log n)$, and this bound is tight.

THEOREM 1. *The maximum number of times that the same angle $0 < \alpha < \pi$ can occur among the ordered triples of $n$ points in the plane is $O(n^2 \log n)$. Furthermore, there are infinitely many values $\alpha$, for which there exists a constant $c(\alpha) > 0$ and $n$-element point sets with the property that at least $c(\alpha)\, n^2 \log n$ triples of them determine angle $\alpha$ (for every $n > 3$).*

We prove the upper bound in Section 2, and the lower bound in Section 3. In Section 4 we consider several related problems, in which we seek upper bounds on the number of triangles spanned by $n$ points in the plane and satisfying a certain common property, for example, having the same area, or having the same perimeter, or being isosceles, etc. We show, by applying and adapting recent results about incidences between points and curves in the plane, that the number of such triangles is at most $O(n^{7/3})$, for a fairly broad class of properties like those just mentioned. However, we do not have a matching lower bound for any of these problems, and we strongly suspect that for all of them the bound is close to quadratic in $n$. Related problems have also been discussed in [14–16, 1]. Our Theorem 1 is one of the rare examples of an exact result in this field.

## 2. THE UPPER BOUND

In this section we shall prove the upper bound in the special case when $\alpha = \pi/2$. That is, we show that the number of right-angle triangles spanned by a set $S$ of $n$ points in the plane is $O(n^2 \log n)$. The proof in the general case is exactly the same as for $\alpha = \pi/2$, except that the exposition is somewhat more complicated, for we can no longer make use of the horizontal and vertical axes of a rectilinear system of coordinates, as we do below.

Let $S$ be a set of $n$ points in the plane. Fix an orientation $\theta$, and consider only those right-angle triangles spanned by points of $S$ which have one side at orientation $\theta$ or $\theta+\pi$ and another side orthogonal to the first side. Without loss of generality assume $\theta=0$.

To bound the number of such triangles from above, we draw a horizontal line $h^{(p)}$ and a vertical line $v^{(p)}$ through each point $p \in S$, thereby obtaining an axis-parallel grid. Every right-angle triangle of the desired form is spanned by a point $p$ in $S$, another point on $h^{(p)}$, and another point on $v^{(p)}$.

Let $h_1, ..., h_k$ denote the horizontal lines of our grid, and let $v_1, ..., v_l$ denote the vertical lines. Put

$$a_i = |S \cap h_i|, \qquad i = 1, ..., k$$
$$b_j = |S \cap v_j|, \qquad j = 1, ..., l.$$

The total number of right-angle triangles of the desired form is therefore at most

$$\sum_{h_i \cap v_j \in S} a_i b_j.$$

We rewrite this sum as

$$\underbrace{\sum_{\substack{h_i \cap v_j \in S \\ a_i > \sqrt{n}}} a_i b_j}_{\sigma_1} + \underbrace{\sum_{\substack{h_i \cap v_j \in S \\ a_i \leqslant \sqrt{n},\, b_j > \sqrt{n}}} a_i b_j}_{\sigma_2} + \underbrace{\sum_{\substack{h_i \cap v_j \in S \\ a_i, b_j \leqslant \sqrt{n}}} a_i b_j}_{\sigma_3}.$$

But

$$\sigma_1 = \sum_{a_i > \sqrt{n}} a_i \cdot \left( \sum_{h_i \cap v_j \in S} b_j \right) \leqslant n \sum_{a_i > \sqrt{n}} a_i,$$

and, similarly,

$$\sigma_2 \leqslant n \sum_{b_j > \sqrt{n}} b_j.$$

Finally, using the Cauchy–Schwartz inequality,

$$\sigma_3 \leqslant \left[ \sum_{\substack{h_i \cap v_j \in S \\ a_i \leqslant \sqrt{n}}} a_i^2 \right]^{1/2} \cdot \left[ \sum_{\substack{h_i \cap v_j \in S \\ b_j \leqslant \sqrt{n}}} b_j^2 \right]^{1/2}. \tag{1}$$

The first summation in the right-hand side of (1) can be rewritten as

$$\sum_{a_i \leqslant \sqrt{n}} \sum_{h_i \cap v_j \in S} a_i^2 = \sum_{a_i \leqslant \sqrt{n}} a_i^2 \cdot |S \cap h_i| = \sum_{a_i \leqslant \sqrt{n}} a_i^3,$$

and, similarly, the second summation is $\sum_{b_j \leqslant \sqrt{n}} b_j^3$. Hence the number of triangles under consideration is at most

$$\left[\sum_{a_i \leqslant \sqrt{n}} a_i^3\right]^{1/2} \cdot \left[\sum_{b_j \leqslant \sqrt{n}} b_j^3\right]^{1/2} + n \sum_{a_i > \sqrt{n}} a_i + n \sum_{b_j > \sqrt{n}} b_j.$$

We remark that the above analysis is a little sloppy, in the sense that the initial bound on the number of desired triangles should have been $\sum_{h_i \cap v_j \in S} (a_i - 1)(b_j - 1)$. In particular, this means that we can (and indeed we do) ignore lines that contain just one point of $S$.

We now repeat the above analysis for every orientation $\theta$ for which there exists a line with orientation $\theta$ that passes through at least two points of $S$. Let $\Theta$ denote the set of all such orientations. Clearly $|\Theta| < n^2$. It follows that the total number, $N$, of right-angle triangles spanned by $S$ satisfies

$$N \leqslant \sum_{\theta \in \Theta} \left\{ \left[\sum_{a_i^\theta \leqslant \sqrt{n}} (a_i^\theta)^3\right]^{1/2} \cdot \left[\sum_{b_j^\theta \leqslant \sqrt{n}} (b_j^\theta)^3\right]^{1/2} + n \sum_{a_i^\theta \leqslant \sqrt{n}} a_i^\theta + n \sum_{b_j^\theta \leqslant \sqrt{n}} b_j^\theta \right\},$$

where $a_i^\theta$ is the number of points of $S$ on the $i$th line of orientation $\theta$ that passes through at least two points of $S$, and $b_j^\theta$ is the number of points of $S$ on the $j$th line of orientation $\theta + \pi/2$ that passes through at least two points of $S$. For each $\theta \in \Theta$, put

$$\alpha_\theta = \left[\sum_{a_i^\theta \leqslant \sqrt{n}} (a_i^\theta)^3\right]^{1/2}$$

and

$$\beta_\theta = \left[\sum_{b_j^\theta \leqslant \sqrt{n}} (b_j^\theta)^3\right]^{1/2}.$$

Then we have

$$\sum_{\theta \in \Theta} \alpha_\theta \beta_\theta \leqslant \left[\sum_{\theta \in \Theta} \alpha_\theta^2\right]^{1/2} \cdot \left[\sum_{\theta \in \Theta} \beta_\theta^2\right]^{1/2},$$

so

$$N \leqslant \left[\sum_{\theta \in \Theta} \alpha_\theta^2\right]^{1/2} \cdot \left[\sum_{\theta \in \Theta} \beta_\theta^2\right]^{1/2} + n \sum_{\theta \in \Theta} \left[\sum_{a_i^\theta \leqslant \sqrt{n}} a_i^\theta + \sum_{b_j^\theta \leqslant \sqrt{n}} b_j^\theta\right].$$

Let $\mathscr{L}$ denote the set of all lines that pass through at least two points of $S$. Let

$$\mathscr{L}_k = \{l \in \mathscr{L} : |l \cap S| = k\}$$
$$\mathscr{L}_{(\geqslant k)} = \{l \in \mathscr{L} : |l \cap S| \geqslant k\}$$
$$\mathscr{L}_{(\leqslant k)} = \{l \in \mathscr{L} : |l \cap S| \leqslant k\}.$$

Note that

$$\sum_{\theta \in \Theta} \alpha_\theta^2, \sum_{\theta \in \Theta} \beta_\theta^2 = \sum_{l \in \mathscr{L}_{(\leqslant \sqrt{n})}} |l \cap S|^3,$$

and, similarly,

$$\sum_{\theta \in \Theta} \left[ \sum_{a_i^\theta > \sqrt{n}} a_i^\theta + \sum_{b_j^\theta > \sqrt{n}} b_j^\theta \right] \leqslant 2 \sum_{l \in \mathscr{L}_{(\geqslant \sqrt{n})}} |l \cap S|,$$

so we can write

$$N \leqslant \sum_{l \in \mathscr{L}_{(\leqslant \sqrt{n})}} |l \cap S|^3 + 2n \sum_{l \in \mathscr{L}_{(\geqslant \sqrt{n})}} |l \cap S|.$$

Put $c_k = |\mathscr{L}_k|$, $C_k = |\mathscr{L}_{(\geqslant k)}|$, for $k \geqslant 2$. Plainly, $c_k = C_k - C_{k+1}$. The results of [22, 4] imply that

$$C_k = O\left(\max\left\{\frac{n^2}{k^3}, \frac{n}{k}\right\}\right).$$

We have

$$N \leqslant \sum_{k \leqslant \sqrt{n}} k^3 c_k + 2n \sum_{k \geqslant \sqrt{n}} k c_k.$$

Rearranging the terms in the first sum, we obtain

$$\sum_{k \leqslant \sqrt{n}} k^3 (C_k - C_{k+1}) = O\left(\sum_{k \leqslant \sqrt{n}} k^2 C_k\right).$$

But for $k \leqslant \sqrt{n}$ we have $n/k \leqslant n^2/k^3$, so we must have $C_k = O(n^2/k^3)$, which implies that the above sum is bounded by

$$O\left(\sum_{k \leqslant \sqrt{n}} \frac{n^2}{k}\right) = O(n^2 \log n).$$

Similarly, the second sum in the bound for $N$ can be rewritten as

$$\sum_{k \geqslant \sqrt{n}} k(C_k - C_{k+1}) = \sqrt{n} C_{\sqrt{n}} + O\left(\sum_{k \geqslant \sqrt{n}} C_k\right).$$

For this range of $k$ we have $n^2/k^3 \leqslant n/k$, so we have $C_k = O(n/k)$, which implies that the sum can be bounded by

$$O\left(\sum_{k \geqslant \sqrt{n}} \frac{n}{k}\right) = O(n \log n).$$

Hence, putting everything together, we obtain

$$N = O(n^2 \log n)$$

as asserted. ∎

## 3. The Lower Bound

In this section we show that the upper bound we have just derived is tight in the worst case. That is, for many values of $\alpha$ we shall construct sets $S$ of $n$ points in the plane such that $\Omega(n^2 \log n)$ triples of them determine angle $\alpha$.

Let $0 < \alpha < \pi$ have the property that $\tan(\alpha) = a\sqrt{m}/b$ for some positive integers $a$, $b$, $m$, such that $m$ is not a square. Let $C(\alpha) = 2\max(am, b)$, and let $n$ be large. Assume further without loss of generality that $n = (4k+1)^2$ for some positive integer $k$. We define $S$ to be the set of all "lattice points" of the form

$$\{(i, j\sqrt{m}) \colon -2k \leqslant i, j \leqslant +2k\}.$$

To obtain a lower bound for the number of all ordered triples of $S$ that determine angle $\alpha$, let $p$ and $q$ be relatively prime integers, $0 < p \leqslant q \leqslant k$, and consider the ray $\rho$ from the origin $O = (0, 0)$ to $(p, q\sqrt{m})$. The slope of $\rho$ is $q\sqrt{m}/p$, and $\rho$ contains at least $[k/\max(p, q)] = [k/q]$ points of $S$ distinct from the origin (here $[x]$ denotes the integer part of $x$). Let $\rho'$ denote the ray from $O$ to $(p', q'\sqrt{m})$ for some relatively prime integers $p'$, $q'$, $|p'| \leqslant k$, $|q'| \leqslant k$. Obviously, $\rho'$ can be obtained from $\rho$ by a counterclockwise rotation with $\alpha$ if and only if

$$\frac{q'\sqrt{m}/p' - q\sqrt{m}/p}{1 + qq'm/pp'} = \tan(\alpha) = a\sqrt{m}/b;$$

i.e.,

$$p'/q' = (pb - qam)/(qb + pa).$$

If this holds, then

(i) there are at least

$$[k/\max(pb-qam, qb+pa)] \geqslant [k/(2\max(p,q)\max(am,b))] = [k/(qC(\alpha))]$$

points of $S$ on $\rho'$; and

(ii) the angle $xOx'$ is equal to $\alpha$ for all $x \in \rho \cap S - \{O\}$, $x' \in \rho' \cap S - \{O\}$.

Exactly the same argument can be repeated for the rays emanating from any lattice point $(i, j\sqrt{m})$, $|i| \leqslant k$, $|j| \leqslant k$. Hence, the number of those triples of $S$ that determine angle $\alpha$ is at least

$$(2k+1)^2 \sum_{q \leqslant k} [k/q][k/(qC(\alpha))]\phi(q) > \frac{n^2}{100C(\alpha)} \cdot \sum_{q \leqslant k} \frac{\phi(q)}{q^2},$$

where $\phi(q)$ denotes the Euler function, i.e., the number of positive integers relatively prime to $q$ (see [17] for basic facts concerning this function). However, it is well known that the last sum is $\Theta(\log k)$. Here is a short proof of this:

$$\sum_{q \leqslant k} \frac{\phi(q)}{q^2} = \sum_{q \leqslant k} \frac{1}{q} \sum_{d \mid q} \frac{\mu(d)}{d},$$

where $\mu(d)$ denotes the Möbius function [17]. Thus this is equal to

$$\sum_{d \leqslant k} \frac{\mu(d)}{d^2} \sum_{i=1}^{[k/d]} \frac{1}{i} \geqslant \sum_{d \leqslant k} \frac{\mu(d)}{d^2} \log \frac{k}{d} - c_1$$

$$\geqslant \log k \sum_{d \leqslant k} \frac{\mu(d)}{d^2} - c_2$$

for appropriate positive constants $c_1, c_2$. But $\sum_{d=1}^{\infty} (\mu(d)/d^2) = 6/\pi^2$, so if $k$ is sufficiently large the expressions above are $\Omega(\log k)$. Indeed,

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \cdot \sum_{i=1}^{\infty} \frac{1}{i^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{d \mid m} \mu(d) = \frac{1}{1^2} \cdot 1 = 1,$$

which implies our claim. Hence $S$ defines at least $c(\alpha)\, n^2 \log n$ triples that span an angle for an appropriate constant $c(\alpha)$. ∎

## 4. Related Problems

In this section we somewhat sharpen and generalize two related results by Erdős and Purdy [14–16]. In all cases we consider a set $S$ of $n$ points in the plane and seek bounds on the maximum number of triangles that are spanned by $S$ and satisfy some common property. The upper bounds we have for all these problems are $O(n^{7/3})$, but none of them is known to be tight. We strongly suspect that the true bounds are not far from being quadratic, as in the case of the problem of repeated angles.

### 4.1. *Repeated Area*

Here we wish to bound the number of triangles that have the same area $A$. Our analysis follows that of Erdős and Purdy. We fix one point $p$ of $S$. For each other point $q$ of $S$ we define a pair of lines $l_q$, $l'_q$ parallel to the segment $pq$ and at distance $2A/|pq|$ from it. Clearly any point $z \in S$ that lies on one of these lines defines a triangle $pqz$ whose area is $A$. Thus the number of triangles that are spanned by $p$ and two other points of $S$ and have area $A$ is half the number of incidences between points in $S - \{p\}$ and the $2(n-1)$ lines $l_q$, $l'_q$, for $q \in S - \{p\}$ (it is half that number because every such triangle is counted twice in the above argument). By the results of [22, 4], the maximum number of such incidences is $O(n^{4/3})$. Repeating this analysis for every point $p \in S$, we obtain

THEOREM 2. *The number of triangles spanned by three points of $S$ and having a specific area $A$ is $O(n^{7/3})$.*

### 4.2. *Isosceles Triangles*

Next consider the problem of bounding the number of isosceles triangles spanned by $S$. The analysis proceeds in much the same way as in the previous subsection. That is, fix a point $p \in S$. For each point $q \in S$ define a circle $C_q$ with center at $q$ and radius $pq$. Any third point $z \in S \cap C_q$ defines an isosceles triangle $pqz$, so that $p$ is one of its base vertices. Thus we need to bound the number of incidences between $n-1$ points and $n-1$ circles, all passing through $p$. Applying an inversion to the plane with $p$ as center, these circles all become lines, so again using the results of [22, 4] we obtain an overall bound of $O(n^{7/3})$. That is,

THEOREM 3. *The number of isosceles triangles spanned by three points of $S$ is $O(n^{7/3})$.*

*Remark.* If one could indeed improve the above bound to $O(n^{2+\varepsilon})$, this would imply, using a standard counting argument (see [4]) that the minimum number of distinct distances between $n$ points in the plane is $\Omega(n^{1-\varepsilon})$, which was conjectured by Erdős in [9, 10]; the currently best lower bound on this quantity is $\Omega(n^{4/5-\varepsilon})$ [8].

*Remark.* Exactly the same analysis shows that the number of triangles spanned by three points of $S$ and having a fixed ratio between two of their sides is also $O(n^{7/3})$.

4.3. *A Note on Incidences between Points and Curves*

The problem of bounding the maximum possible number of incidences between $m$ points and $n$ curves of certain kinds in the plane has been studied in several papers that we have already cited [22, 21, 4]. For certain special families of curves better upper bounds can be obtained—for lines (and pseudo-lines) and for unit circles the bound is $O(m^{2/3}n^{2/3}+m+n)$ and for arbitrary circles (or pseudo-circles) the bound is $O(m^{4/5}n^{4/5}+m+n)$. A "default" weaker upper bound can be obtained as follows. Let $G$ be the bipartite graph $(V_1 \cup V_2, E)$, where $V_1$ is the set of points, $V_2$ is the set of curves, and $E$ contains edges of the form $(p, c)$ is point $p$ lies on curve $c$. If we assume that no pair of the given curves intersect in more than $s$ points, for some fixed constant $s$, then $G$ does not contain $K_{s+1,2}$ as a subgraph, so by standard extremal graph-theoretic arguments we must have $|E| = O(m^{1/2}n+m)$ (where the constant of proportionality depends on $s$)

In this subsection we obtain better bounds for fairly general collections of curves. We assume as before that each pair of curves intersect in at most a constant number of points, but in addition we also assume that each curve in the collection can be defined in terms of $d$ real parameters, in the sense that any assignment of values to these parameters can define at most a constant number of curves. Moreover, we assume that the dependence of the curves on these parameters is algebraic of low degree, so that the following slightly stronger property holds. For any $d$ given points, the number of curves that pass through all of them is at most another constant $s$. For example, $d=2$ for lines and for unit circles, and $d=3$ for arbitrary circles, and the stronger property is satisfied in all these cases. Our result is

THEOREM 4. *Under the above assumptions, the maximum number of incidences between m points and n curves is*

$$O(m^{d/(2d-1)}n^{(2d-2)/(2d-1)}+m+n).$$

*Proof.* We follow the analysis given in [4]. First consider the bipartite graph $G$ as defined above and observe that our assumptions imply that $G$ does not contain $K_{d,s+1}$ as a subgraph, so again by extremal graph-theoretic arguments, the maximum number of incidences between the points and curves is $O(mn^{(d-1)/d}+n)$. This is the so called *Canham bound* or *Canham threshold* in [4].

Next we choose a random sample $R$ of $r$ of the $n$ given curves and form the arragement $\mathscr{A}(R)$ of the sample curves. We partition each face of $\mathscr{A}$ further, using the *vertical decomposition* technique described in [4]. We obtain a collection of $O(r^2)$ trapezoidal-like subcells, which we call *funnels*. Let $m_i$, $n_i$ denote respectively the number of points lying in, and the number of curves passing through, the interior of the $i$th funnel (if a point lies on a vertical boundary edge, we assign it to both adjacent funnels). The number of incidences between those subsets of points and curves is $\sum_i O(m_i n_i^{(d-1)/d} + n_i)$. What we still miss are incidences with points that lie on one of the sample curves. But the maximum number of incidences that any of the $n$ curves can form with points on the sample curves is $O(r)$ (because it intersects each sample curve in a constant number of points). This captures most missing incidences, except possibly those involving points that lie on just one sample curve, and their number is clearly at most $m$. Thus the number of missing incidences is at most $O(m + nr)$. Using the results of Clarkson and Shor [5], the expected value of $\sum_i m_i n_i^{(d-1)/d}$ is $O(m(n/r)^{(d-1)/d})$, and the expected value of $\sum_i n_i$ is $O(nr)$. Putting it all together, the number of incidences is at most

$$O\left(m\left(\frac{n}{r}\right)^{(d-1)/d} + nr + m\right).$$

Choosing $r = m^{d/(2d-1)}/n^{1/(2d-1)}$ we obtain the desired bound. (This can be done provided $m^d \geqslant n$. Otherwise the Canham bound itself becomes $O(n)$.) ∎

4.4. *Repeated Perimeter*

We next give an application of the result just derived. We wish to bound the number of triangles spanned by points of $S$ and having a specific perimeter $P$. Using the same approach as above, we fix a point $p \in S$ and, for each other point $q \in S$, we consider the locus $\gamma_q$ of points $z$ such that $|pz| + |qz| = P - |pq|$. Clearly this is an ellipse having $p$ and $q$ as foci. We now need to bound the number of incidences between $n-1$ such ellipses and $n-1$ points. Now we can apply the results of the preceding subsection. Indeed, each of these ellipses is defined in terms of two real parameters, that is by specifying the second focus $q$—since $P$ is common to all those ellipses, specification of $q$ uniquely determines the ellipse. The preceding results thus imply that the number of incidence in question is $O(n^{4/3})$, and we thus obtain

THEOREM 5. *The number of triangles spanned by three points of $S$ and having a specific perimeter $P$ is $O(n^{7/3})$.*

*Remark.* The last argument is general enough to yield a bound of $O(n^{7/3})$ on the number of triangles spanned by $S$ and having any reasonable specific property, such as having a specific angle bisector, having a specific difference between two of their angles, having an incircle radius, etc. We leave it to the interested reader to verify the details of such generalizations.

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